Kinetic Equations

Solution to the Exercises

– 27.05.2021 –

Teachers: Prof. Chiara Saffirio, Dr. Théophile Dolmaire

Assistant: Dr. Daniele Dimonte – `daniele.dimonte@unibas.ch`

Exercise 1

Let $f : [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be a function which is $\mathcal{C}^1$ in t and x, such that $(\omega, v_*) \mapsto B(v - v_*, \omega)\big(f' f'_* - f f_*\big)$ is integrable for all (t,x,v).

(i) Assume that f solves the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \tag{1}$$

with initial datum f_0. Prove that

$$f(t,x,v) = f_0(x - tv, v) + \int_0^t Q(f,f)(s, x - (t-s)v, v) ds. \tag{2}$$

In the following we will call a continuous function f which is solution of (2) a *mild solution* of the Boltzmann equation.

We consider now a system of *Maxwellian molecules*, i.e. a system in which B is of the form $B(v - v_*, \omega) = b(\cos\theta)$, where we indicate with θ the angle between ω and the vector $v - v_*$. On b we only assume that $\int_{\mathbb{S}^2} b(\cos\theta)\, d\omega$ is finite and bounded uniformly in v and v_* (notice that by definition θ depends on ω and v_*), i.e. there exists a positive real number β such that

$$\int_{\mathbb{S}^2} b(\cos\theta)\, d\omega \leqslant \beta, \qquad \forall\, (v, v_*) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{3}$$

We also call $\varphi(v) := e^{-\alpha|v|^2}$, with $\alpha > 0$, a Maxwellian function and $M := \int_{\mathbb{R}^3} \varphi(v)\, dv$ its mass.

Finally, assume that $f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function such that $|f_0| \leqslant \varphi$ and define the sequence of functions $\left\{\widetilde{f}_n\right\}_{n \geqslant 0}$ defined recursively as

$$\begin{cases} \widetilde{f}_0(t,x,v) = f_0(x,v), \\ \widetilde{f}_{n+1}(t,x,v) = f_0(x - tv, v) + \int_0^t Q(\widetilde{f}_n, \widetilde{f}_n)(s, x - (t-s)v, v) ds. \end{cases} \tag{4}$$

(ii) Assuming that $|\widetilde{f}_n(t,x,v)| \leqslant 2\varphi(v)$, prove that $|\widetilde{f}_{n+1}(t,x,v)| \leqslant (1 + 8\beta M t)\varphi(v)$, for all $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$, and all $n \geqslant 0$. Prove also that $\widetilde{f}_n$ is continuous, for all $n \geqslant 0$.

(iii) We define $T = 1/(8\beta M)$. Prove that, for all $t \in [0,T]$ and $n \geqslant 0$,

$$|\widetilde{f}_n(t,\cdot,\cdot)| \leqslant 2\varphi. \tag{5}$$

(iv) Define

$$\|f\|_{\varphi} := \sup_{(t,x,v)\in[0,T]\times\mathbb{R}^3\times\mathbb{R}^3} \frac{|f(t,x,v)|}{\varphi(v)}, \tag{6}$$

where it's important to notice that now the supremum in t is taken on the interval $[0,T]$.

Denote $[f](v,v_*,\omega) = f(v')f(v'_*) - f(v)f(v_*)$ (where we omit everywhere the variable x out of convenience) and prove that, for all $x, v, v_* \in \mathbb{R}^3$, $t \in [0,T]$, $\omega \in \mathbb{S}^2$ and all $n \geqslant 0$:

$$\frac{\left|[\widetilde{f}_{n+1}](v,v_*) - [\widetilde{f}_n](v,v_*)\right|}{\varphi(v)\varphi(v_*)} \leqslant 8\|\widetilde{f}_{n+1} - \widetilde{f}_n\|_{\varphi}. \tag{7}$$

(v) Use (7) to bound $|Q(\widetilde{f}_{n+1},\widetilde{f}_{n+1})(v) - Q(\widetilde{f}_n,\widetilde{f}_n)(v)|$ and to deduce that for all $t \in [0,T]$

$$\|Q(\widetilde{f}_{n+1},\widetilde{f}_{n+1}) - Q(\widetilde{f}_n,\widetilde{f}_n)\|_{\varphi} \leqslant 8\beta M\|\widetilde{f}_{n+1} - \widetilde{f}_n\|_{\varphi}. \tag{8}$$

(vi) Prove, for all $n \geqslant 1$, that

$$\begin{cases} \|\widetilde{f}_{n+1} - \widetilde{f}_n\|_{\varphi} \leqslant 8\beta MT\|\widetilde{f}_n - \widetilde{f}_{n-1}\|_{\varphi}, \\ \|Q(\widetilde{f}_{n+1},\widetilde{f}_{n+1}) - Q(\widetilde{f}_n,\widetilde{f}_n)\|_{\varphi} \leqslant 8\beta MT\|Q(\widetilde{f}_n,\widetilde{f}_n) - Q(\widetilde{f}_{n-1},\widetilde{f}_{n-1})\|_{\varphi}. \end{cases} \tag{9}$$

Deduce, for any $0 < \alpha < 1$, that the sequences of functions $\{\widetilde{f}_n\}_{n\geqslant 0}$ and $\left\{Q(\widetilde{f}_n,\widetilde{f}_n)\right\}_{n\geqslant 0}$ are respectively converging uniformly towards some continuous limits f and $\widetilde{Q}$ on $[0,\alpha T]\times\mathbb{R}^3\times\mathbb{R}^3$.

(vii) Prove that f is a mild solution of the Boltzmann equation with initial datum f_0.

Remark. *We recall that, for particles interacting via inverse-power laws potentials $\phi(r) = 1/r^{k-1}$ (with $k > 2$), the collision kernel $B(v - v_*, \cos\theta)$ takes the particular form $B = b(\cos\theta)|v - v_*|^{\gamma}$, with $\gamma = (k-5)/(k-1)$, and b locally smooth. The case we just considered is the case of Maxwellian molecules, corresponding to the case $\gamma = 0$.*

Proof. To see (i) we first notice that if we differenciate in time the function $f(t,x+tv,v)$ we get

$$\partial_t\left(f(t,x+tv,v)\right) = \left(\partial_t f\right)(t,x+tv,v) + v\cdot\left(\nabla_x f\right)(t,x+tv,v) \tag{10}$$

$$= \left(\partial_t f + v\cdot\nabla_x f\right)(t,x+tv,v) \tag{11}$$

$$= Q(f,f)(t,x+tv,v). \tag{12}$$

Therefore, using the Duhamel principle we get

$$f(t,x+tv,v) = f_0(x,v) + \int_0^t Q(f,f)(s,x+sv,v)\,ds. \tag{13}$$

Applying the last equality at the position $x' = x - tv$ we get (2).

To proof (ii), we get first that

$$|Q(f,f)(x,v)| = \left| \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos\theta)\left(f' f'_* - f f_*\right) d\omega\ dv_* \right| \tag{14}$$

$$\leqslant \left| \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos\theta) f' f'_* d\omega\ dv_* \right| + \left| \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos\theta) f f_* d\omega\ dv_* \right|. \tag{15}$$

Notice now that if we assume that $|f| \leqslant 2\varphi$ we get

$$\left|f' f'_*\right| = \left|f(x,v') f(x,v'_*)\right| \leqslant 4\varphi(v')\varphi(v'_*) = 4e^{-\alpha\left(|v'|+|v'_*|\right)} \tag{16}$$

$$= 4e^{-\alpha(|v|+|v_*|)} = 4\varphi(v)\varphi(v_*), \tag{17}$$

where we used that by definition the transformation $(v,v_*) \mapsto (v',v'_*)$ preserves the kinetic energy. As a consequence if $|f| \leqslant \varphi_)$ we bound $Q(f,f)$ as

$$|Q(f,f)(x,v)| \leqslant 8 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos\theta)\varphi_0(v)\varphi_0(v_*) d\omega\ dv_* \leqslant 8\beta M\varphi(v). \tag{18}$$

Assume now that $\left|\widetilde{f}_n(t,x,v)\right| \leqslant 2\varphi(v)$, we get

$$\left|\widetilde{f}_{n+1}(t,x,v)\right| = \left| f_0(x-tv,v) + \int_0^t Q\left(\widetilde{f}_n,\widetilde{f}_n\right)(s,x-(t-s)v,v)\,ds \right| \tag{19}$$

$$\leqslant \varphi(v) + \int_0^t 8\beta M\varphi(v)\,ds \tag{20}$$

$$\leqslant (1+8\beta Mt)\varphi(v). \tag{21}$$

The continuity can be easily proven with the definition and using induction.

If we now define $T := \frac{1}{8\beta M}$ as in (iii) we get immediately that if $\left|\widetilde{f}_n(t,x,v)\right| \leqslant 2\varphi(v)$ for any $t \in [0,T]$, then $\left|\widetilde{f}_{n+1}(t,x,v)\right| \leqslant (1+8\beta MT)\varphi(v) = 2\varphi(v)$, therefore (5) follows easily by induction.

To prove (iv) we first notice (we omit the t and x variables out of convenience)

$$\left|\left[\widetilde{f}_{n+1}\right](v,v_*) - \left[\widetilde{f}_n\right](v,v_*)\right| = \Big|\widetilde{f}_{n+1}(v')\widetilde{f}_{n+1}(v'_*) - \widetilde{f}_{n+1}(v)\widetilde{f}_{n+1}(v_*) \tag{22}$$

$$-\widetilde{f}_n(v')\widetilde{f}_n(v'_*) + \widetilde{f}_n(v)\widetilde{f}_n(v_*)\Big| \tag{23}$$

$$\leqslant \left|\widetilde{f}_{n+1}(v') - \widetilde{f}_n(v')\right|\left|\widetilde{f}_{n+1}(v'_*)\right| \tag{24}$$

$$+\left|\widetilde{f}_n(v')\right|\left|\widetilde{f}_{n+1}(v'_*) - \widetilde{f}_n(v'_*)\right| \tag{25}$$

$$+\left|\widetilde{f}_{n+1}(v) - \widetilde{f}_n(v)\right|\left|\widetilde{f}_{n+1}(v_*)\right| \tag{26}$$

$$+\left|\widetilde{f}_n(v)\right|\left|\widetilde{f}_{n+1}(v_*) - \widetilde{f}_n(v_*)\right|. \tag{27}$$

We now use that for any $t \in [0,T]$ and for any n we have $\left|\widetilde{f}_n\right| \leqslant 2\varphi$ to get

$$\frac{\left|\left[\widetilde{f}_{n+1}\right](v,v_*) - \left[\widetilde{f}_n\right](v,v_*)\right|}{\varphi(v)\varphi(v_*)} \leqslant 2\frac{\left|\widetilde{f}_{n+1}(v') - \widetilde{f}_n(v')\right|}{\varphi(v')} \tag{28}$$

$$+ 2\frac{\left|\widetilde{f}_{n+1}(v'_*) - \widetilde{f}_n(v'_*)\right|}{\varphi(v'_*)} \tag{29}$$

$$+ 2\frac{\left|\widetilde{f}_{n+1}(v) - \widetilde{f}_n(v)\right|}{\varphi(v)} \tag{30}$$

$$+ 2\frac{\left|\widetilde{f}_{n+1}(v_*) - \widetilde{f}_n(v_*)\right|}{\varphi(v_*)} \tag{31}$$

$$\leqslant 8\left\|\widetilde{f}_{n+1} - \widetilde{f}_n\right\|_\varphi \tag{32}$$

where we also used the fact that we saw before that $\varphi(v')\varphi(v'_*) = \varphi(v)\varphi(v_*)$

To prove (v), we look at the difference $Q\left(\widetilde{f}_{n+1},\widetilde{f}_{n+1}\right) - Q\left(\widetilde{f}_n,\widetilde{f}_n\right)$ to get

$$\frac{\left|Q\left(\widetilde{f}_{n+1},\widetilde{f}_{n+1}\right)(v) - Q\left(\widetilde{f}_n,\widetilde{f}_n\right)(v)\right|}{\varphi(v)} = \tag{33}$$

$$= \left|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\frac{\left[\widetilde{f}_{n+1}\right](v,v_*) - \left[\widetilde{f}_n\right](v,v_*)}{\varphi(v)} d\omega\ dv_*\right| \tag{34}$$

$$\leqslant 8\beta M\left\|\widetilde{f}_{n+1} - \widetilde{f}_n\right\|_\varphi. \tag{35}$$

We now turn our attention to (vi); we get, by definition

$$\left|\widetilde{f}_{n+1}(t,x,v) - \widetilde{f}_n(t,x,v)\right| = \left|\int_0^t \left(Q\left(\widetilde{f}_n,\widetilde{f}_n\right)(s,x-(t-s)v,v)\right.\right. \tag{36}$$

$$\left.\left. -Q\left(\widetilde{f}_{n-1},\widetilde{f}_{n-1}\right)(s,x-(t-s)v,v)\right) ds\right| \tag{37}$$

$$\leqslant t\left\|Q\left(\widetilde{f}_n,\widetilde{f}_n\right) - Q\left(\widetilde{f}_{n-1},\widetilde{f}_{n-1}\right)\right\|_\varphi \varphi(v) \tag{38}$$

$$\leqslant T\left\|Q\left(\widetilde{f}_n,\widetilde{f}_n\right) - Q\left(\widetilde{f}_{n-1},\widetilde{f}_{n-1}\right)\right\|_\varphi \varphi(v). \tag{39}$$

This implies immediately

$$\left\|\widetilde{f}_{n+1}(t,x,v) - \widetilde{f}_n(t,x,v)\right\|_\varphi \leqslant T\left\|Q\left(\widetilde{f}_n,\widetilde{f}_n\right) - Q\left(\widetilde{f}_{n-1},\widetilde{f}_{n-1}\right)\right\|_\varphi. \tag{40}$$

If we now combine (8) with (40) we get (9).

Now, notice that $\|\cdot\|_\varphi$ is a norm, therefore, we can use (9) to prove that there is a limit. Indeed, all the statements above are still true if instead of T we consider αT with $\alpha \in [0,1]$, given that the only case where we use the explicit definition of T was to prove that

$\left|\widetilde{f}_n\right| \leqslant 2\varphi$, but this is also true for αT. We can now fix $\alpha = \frac{1}{2}$ to rewrite (9) as

$$\begin{cases} \left\|\widetilde{f}_{n+1}(t,x,v) - \widetilde{f}_n(t,x,v)\right\|_\varphi \leqslant \frac{1}{2}\left\|\widetilde{f}_n(t,x,v) - \widetilde{f}_{n-1}(t,x,v)\right\|_\varphi \\ \left\|Q\left(\widetilde{f}_{n+1},\widetilde{f}_n\right) - Q\left(\widetilde{f}_n,\widetilde{f}_{n-1}\right)\right\|_\varphi \leqslant \frac{1}{2}\left\|Q\left(\widetilde{f}_n,\widetilde{f}_n\right) - Q\left(\widetilde{f}_{n-1},\widetilde{f}_{n-1}\right)\right\|_\varphi \end{cases} \tag{41}$$

Therefore the sequences $\left\{\widetilde{f}_n\right\}_{n\in\mathbb{N}}$ and $\left\{Q\left(\widetilde{f}_n,\widetilde{f}_n\right)\right\}_{n\in\mathbb{N}}$ are Cauchy sequences for $\|\cdot\|_\varphi$. Given that $\|\cdot\|_\infty \leqslant \|\cdot\|_\varphi$, $\widetilde{f}_n \to f$ and $Q\left(\widetilde{f}_n,\widetilde{f}_n\right) \to \widetilde{Q}$ as $n \to +\infty$ uniformly in $(t,x,v) \in \left[0,\frac{1}{2}\right] \times \mathbb{R}^3 \times \mathbb{R}^3$.

To conclude with the proof of (vii) we notice that

$$f(t,x,v) = \lim_{n\to+\infty} \widetilde{f}_n(t,x,v) \tag{42}$$

$$= f_0(x,t,v) + \lim_{n\to+\infty}\int_0^t Q\left(\widetilde{f}_{n-1},\widetilde{f}_{n-1}\right)(s, x-(t-s)v, v)\,ds \tag{43}$$

$$= f_0(x,t,v) + \int_0^t \widetilde{Q}(s, x-(t-s)v, v)\,ds. \tag{44}$$

On the other hand, if we use the fact that $\sup_n \left|\widetilde{f}_n\right| \leqslant 2\varphi$ to apply the dominated convergence Theorem, we also have

$$\widetilde{Q}(t,x,v) = \lim_{n\to+\infty} Q\left(\widetilde{f}_n,\widetilde{f}_n\right)(t,x,v) \tag{45}$$

$$= \lim_{n\to+\infty}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\left[\widetilde{f}_n\right](t,x,v,v_*)\,d\omega\,dv_* \tag{46}$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)[f](t,x,v,v_*)\,d\omega\,dv_* = Q(f,f)(t,x,v), \tag{47}$$

which tells us that f is indeed a mild solution.

□

Exercise 2

Note that to make sense, a mild solution of the Boltzmann equation, as defined by (2), does not need to be differentiable, with respect to any of its variables.
Using the result of the previous exercise, providing a (local in time) mild solution f to the Boltzmann equation such that $|f| \leqslant 2\varphi$, prove that

$$\lim_{h\to 0}\frac{1}{h}\big(f(t+h, x+hv, v) - f(t,x,v)\big) \tag{48}$$

makes sense for all fixed $(t,x,v) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$.

For a general function f which is $\mathcal{C}^1$ in t and x , what is the limit, when h goes to zero, of the quantity (48)?

Proof. From the fact that f is a mild solution we get

$$
\begin{aligned}
f\left(t+h, x+hv, v\right) &= f_0\left(x+hv-\left(t+h\right)v, v\right) && (49)\\
&\quad + \int_0^{t+h} Q\left(f,f\right)\left(s, x+hv-\left(t+h-s\right)v, v\right) ds && (50)\\
&= f_0\left(x-tv, v\right) && (51)\\
&\quad + \int_0^{t+h} Q\left(f,f\right)\left(s, x-\left(t-s\right)v, v\right) ds. && (52)
\end{aligned}
$$

As a consequence, through the fundamental theorem of calculus we get

$$
\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\left(f\left(t+h,x+hv,v\right)-f\left(t,x,v\right)\right) &= \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} Q\left(f,f\right)\left(s,x-\left(t-s\right)v,v\right) ds && (53)\\
&= Q\left(f,f\right)\left(t,x,v\right). && (54)
\end{aligned}
$$

Now, if f is regular enough, we get

$$
\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\left(f\left(t+h,x+hv,v\right)-f\left(t,x,v\right)\right) = && (55)\\
&= \lim_{h\to 0}\frac{1}{h}\left(f\left(t+h,x+hv,v\right)-f\left(t,x+hv,v\right)\right) && (56)\\
&\quad + \lim_{h\to 0}\frac{1}{h}\left(f\left(t,x+hv,v\right)-f\left(t,x,v\right)\right) && (57)\\
&= \partial_t f\left(t,x,v\right)+v\cdot\nabla_x f\left(t,x,v\right), && (58)
\end{aligned}
$$

therefore if f is regular enough, it is a classical solution of (1)

□

Exercise 3

In the case of hard spheres, the loss term of the Boltzmann equation writes $L(f)(v)f(v)$, where

$$L(f)(v) = \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} |\omega\cdot(v-v_*)| f(v_*) dv_*\, d\omega. \tag{59}$$

Denote now as φ_α the Maxwellian function $e^{-\alpha|v|^2}$.

In their famous article of 1978, Kaniel and Shinbrot introduced the following cathegorization on L: if there exists a positive constant $C(\alpha)$ depending only on α and a positive number $0 \leqslant \lambda < 2$ such that, for all $v \in \mathbb{R}^3$

$$L(\varphi_\alpha)(v) \leqslant C(\alpha)(1+|v|^\lambda), \tag{60}$$

the collision kernel B describes a *soft interaction* if $\lambda = 0$, and it describes a *hard interaction* if $\lambda > 0$.

(i) Show that in the case of the hard spheres, condition (60) holds for $\lambda = 1$, that is one can find a constant $C(\alpha)$ such that (60) holds for all $v \in \mathbb{R}^3$.

(ii) One may wonder if this control can be improved in the case of the hard spheres. Show that we cannot choose $\lambda = 0$ (so that, of course, the hard sphere collision kernel does *not* represent a soft interaction).

(iii) Show that (60) does not hold for any $0 < \lambda < 1$ in the hard sphere case.

Proof. To prove (i) we get

$$
\begin{aligned}
L(\varphi_\alpha)(v) &= \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} |\omega \cdot (v - v_*)|\, \varphi_\alpha(v_*)\, dv_*\, d\omega && (61)\\
&\leqslant |\mathbb{S}^2| \int_{\mathbb{R}^3} |v - v_*|\, \varphi_\alpha(v_*)\, dv_* && (62)\\
&\leqslant |\mathbb{S}^2| \left(\int_{\mathbb{R}^3} |v_*|\, \varphi_\alpha(v_*)\, dv_* + |v| \int_{\mathbb{R}^3} \varphi_\alpha(v_*)\, dv_* \right), && (63)
\end{aligned}
$$

which implies the condition 60 for $\lambda = 1$.

Now, using the change of variable $v - v_* = u$ we get

$$
L(\varphi_\alpha)(v) = \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} |\omega \cdot u|\, e^{-\alpha|v-u|^2} du\, d\omega. \tag{64}
$$

We now apply a rotation to the integral in ω in such a way that u coincide with the vertical axis. If we do so for any $|v| \geqslant 1$ we get

$$
\begin{aligned}
L(\varphi_\alpha)(v) &= \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} |\omega_3|\, |u|\, e^{-\alpha|v-u|^2} du\, d\omega && (65)\\
&= \left(\int_{\mathbb{S}^2} |\omega_3|\, d\omega \right) \int_{\mathbb{R}^3} |u|\, e^{-\alpha|v-u|^2} du && (66)\\
&= \left(\int_{\mathbb{S}^2} |\omega_3|\, d\omega \right) \int_{\mathbb{R}^3} |u + v|\, e^{-\alpha|u|^2} du && (67)\\
&\geqslant \left(\int_{\mathbb{S}^2} |\omega_3|\, d\omega \right) \int_{B_{\frac{|v|}{2}}(0)} |u + v|\, e^{-\alpha|u|^2} du && (68)\\
&\geqslant \left(\int_{\mathbb{S}^2} |\omega_3|\, d\omega \right) \int_{B_{\frac{|v|}{2}}(0)} (|v| - |u|)\, e^{-\alpha|u|^2} du && (69)\\
&\geqslant \left(\int_{\mathbb{S}^2} |\omega_3|\, d\omega \right) \frac{|v|}{2} \int_{B_{\frac{|v|}{2}}(0)} e^{-\alpha|u|^2} du && (70)\\
&\geqslant \left(\int_{\mathbb{S}^2} |\omega_3|\, d\omega \int_{B_{\frac{1}{2}}(0)} e^{-\alpha|u|^2} du \right) \frac{|v|}{2}. && (71)
\end{aligned}
$$

This implies that 60 cannot be true for any $\lambda \in [0, 1)$.

□